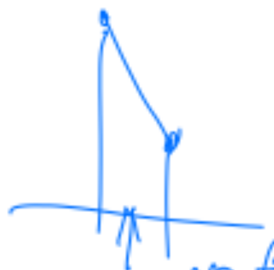
On 3.38 ~

(b) Where was $g'(x)$ least (largest negative #)

Look for: largest drop in $g(x)$ values.



good guess for x value

(c) Prob no place where $g'(x) = 0$, because $g(x)$ is decreasing on the interval.

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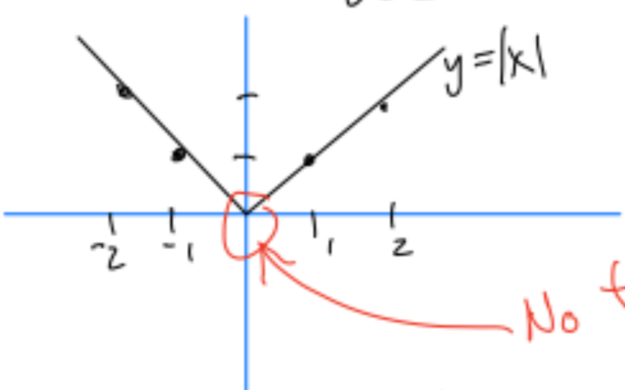
Examples where the derivative does not exist:

(1) $y = |x|$ ← the derivative does not exist at $x = 0$.




No tangent line: When you zoom in, it does not look like a line.

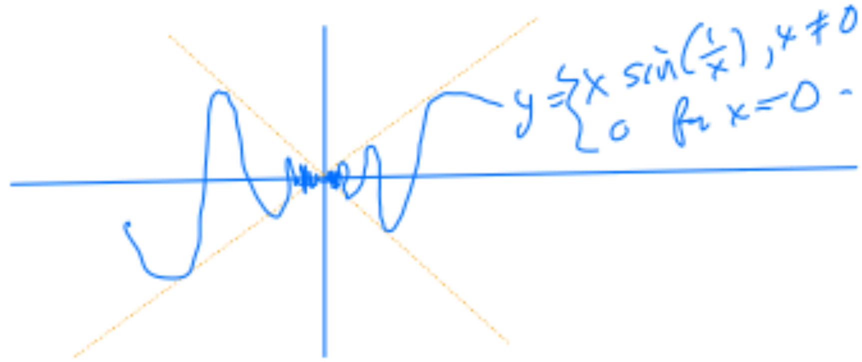
⇒ Not differentiable at $x = 0$.

② $$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

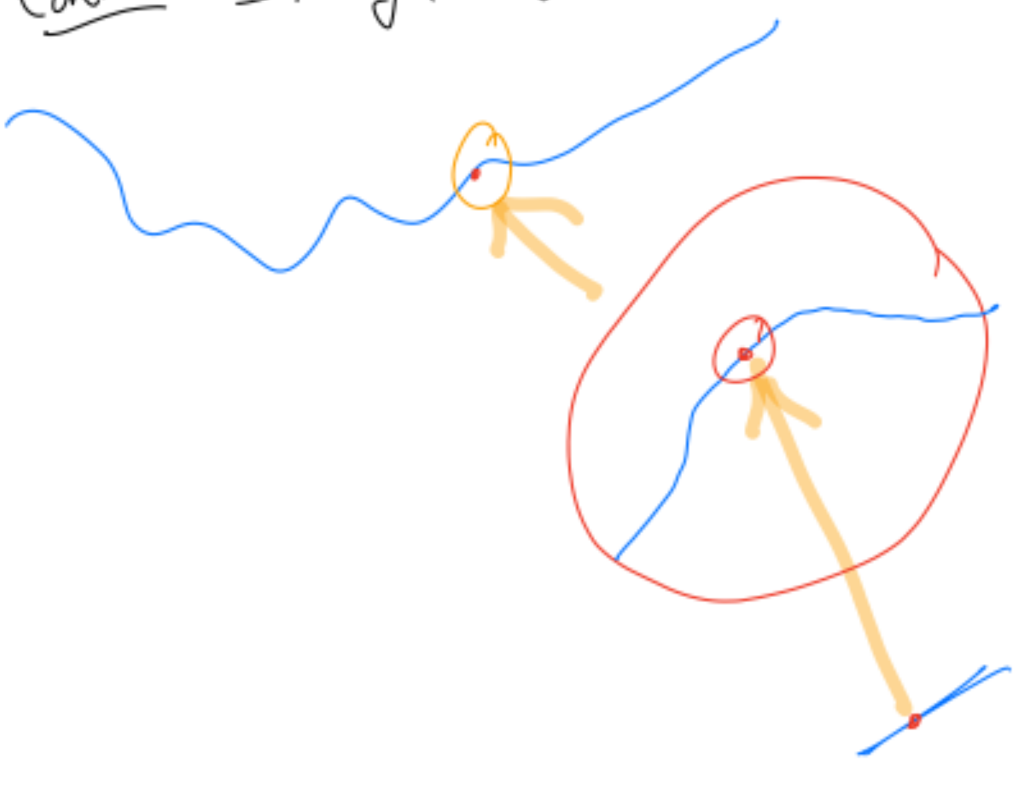
$f(x)$ is continuous at every $x \in \mathbb{R}$

$f(x)$ is differentiable at every $x \neq 0$.

<u>Not</u> differentiable at $x = 0$.




<u>Contrast</u>: If $g(x)$ is differentiable at $a$



# More tools for taking derivatives fast:

- Chain Rule

If $f$ & $g$ are differentiable functions, then

$$\underbrace{(f \circ g)'(x)}_{\circ\ \text{composition}} = \Big(f(g(x))\Big)'$$

$$\boxed{\Big(f(g(x))\Big)' = f'(g(x)) \cdot g'(x).}$$

chain rule

Example: $Q(x) = \sin(x^2)$

$f(x) = \sin(x)$

$g(x) = x^2$

$$f \circ g(x) = f(g(x))$$
$$= \sin(x^2)$$

Chain Rule:

$$\Big(f(\text{Bubba})\Big)' = f'(\text{Bubba}) \cdot (\text{Bubba})'$$

Proof of the Chain Rule

$$\Big(f(g(x))\Big)' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

How do we compute this?

Alternate defn of derivative

$$F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}$$

same as before, but with a+h replaced by x.

$$\Rightarrow \left(f(g(a))\right)' = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x\to a} \underbrace{\frac{f(g(x)) - f(g(a))}{(g(x) - g(a))}}_{f'(g(a))} \underbrace{\frac{(g(x) - g(a))}{(x-a)}}_{g'(a)}$$

$$= f'(g(a))\cdot g'(a)$$

---

## Examples

① $F(t) = \sin(t^2)$. Find $F'(t)$.

$$F'(t) = \underbrace{\cos}_{(\sin')}(t^2)\cdot(t^2)'$$

$$= \cos(t^2)\cdot(2t) = \boxed{2t\cos(t^2)}$$

(2) $G(x) = \sqrt{2x-5}$

Find $\frac{dG}{dx}$.

Solution: $\left(\begin{array}{l}\text{outside function} = \sqrt{x} = x^{1/2} \\ \text{inside function} = 2x-5\end{array}\right.$

$$\frac{dG}{dx} = \frac{1}{2}(2x-5)^{-1/2} \cdot (2x-5)'$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2x-5}} \cdot 2 = \boxed{\frac{1}{\sqrt{2x-5}}}$$

(3) $p(s) = 7 \cdot 3^{4s}$ $\quad p'(s) = ?$

Solution: $p'(s) = (7 \cdot 3^{4s})'$

$$= 7(3^{4s})' = 7\left(\left(e^{\ln 3}\right)^{4s}\right)'$$

$$= 7\left(e^{(\ln 3)4s}\right)'$$

outside fun: $e^x$

inside fun: $(\ln 3)4s$

$$= 7e^{(\ln 3)4s} \cdot \left((\ln 3)4s\right)'$$

$$= 7\underbrace{e^{(\ln 3)4s}}_{3^{4s}} \cdot (\ln 3)4.$$

$$= \boxed{(28 \cdot \ln 3)3^{4s}}$$

---

Formula for $(a^x)' = ?$

$a$ is a #.

$$(a^x)' = \left(\left(e^{(\ln a)}\right)^x\right)'$$

$$= \left(e^{(\ln(a)x)}\right)' = \underbrace{e^{(\ln a)x}}_{a^x} \cdot \ln(a)$$

$$\boxed{(a^x)' = (\ln(a))a^x}$$

# Inverse function Theorem.

$f$ and $g$ are called inverse functions if $f(g(x)) = x$

and $g(f(x)) = x$

---

example: $\sqrt[3]{x}$ & $x^3$ are inverses

since $\left(\sqrt[3]{x}\right)^3 = x$, $\sqrt[3]{x^3} = x$

---

partial inverses:

$\sqrt{x}$ & $x^2$

$\left(\sqrt{x}\right)^2 = x$

$\sqrt{x^2} = x$ but only if $x > 0$.

---

$\sin(\arcsin(x)) = x$.

If $f$ & $g$ are inverse

$$f(g(x)) = x$$

Take derivative of both sides

$$\big(f(g(x))\big)' = (x)'$$

$$f'(g(x))\, g'(x) = 1$$

$$\Rightarrow \boxed{g'(x) = \frac{1}{f'(g(x))}} \quad \text{IFT}$$

inverse function derivative formula.

Example: $f(x) = e^x$, $g(x) = \ln(x)$

$$f(g(x)) = e^{\ln x} = x.$$

$$\text{IFT} \Rightarrow g'(x) = (\ln(x))' = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}}$$

$$\Rightarrow g'(x) = \frac{1}{x}$$

$$\Rightarrow \boxed{\left(\ln(x)\right)' = \frac{1}{x}}$$

Another example of the chain rule:

Find the derivative of $\cos^2\left(e^{\sin(\ln x)}\right)$.

Solution:

$\cos^2 = x^2 \circ \cos$

outside $= x^2$

inside $= \cos\left(e^{\sin(\ln x)}\right)$

$$\left[\cos^2\left(e^{\sin(\ln x)}\right)\right]' =$$

$$2\cos\left(e^{\sin(\ln x)}\right) \cdot \left[\cos\left(e^{\sin(\ln x)}\right)\right]'$$

$$= 2\cos\left(e^{\sin(\ln x)}\right)\left(-\sin\left(e^{\sin(\ln x)}\right) \cdot \left[e^{\sin(\ln x)}\right]'\right.$$

outside $= e^x$

$$= -2\cos\left(e^{\sin(\ln x)}\right)\sin\left(e^{\sin(\ln x)}\right) \cdot e^{\sin(\ln x)} \cdot \left[\sin(\ln x)\right]'$$

sin = outside

$$= -2\cos\left(e^{\sin(\ln x)}\right)\sin\left(e^{\sin(\ln x)}\right)\cdot e^{\sin(\ln x)}\cos(\ln x)\cdot[\ln x]'$$

$$= -2\cos\left(e^{\sin(\ln x)}\right)\sin\left(e^{\sin(\ln x)}\right)\cdot e^{\sin(\ln x)}\cdot\cos(\ln x)\,\frac{1}{x}$$